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EE225D

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# Digital Filters

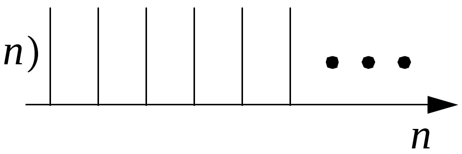
# Lecture 7

# 1. Example of inverse z-transform use.

- Let input be $u(n)$ and filter be $y(n) = ay(n-1) + x(n)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-1}$$

$x(n) = u(n)$




$$Y(z) = \frac{1}{1 - az^{-1}} \qquad \text{so} \qquad y(n) = \frac{1}{2\pi j}\oint \frac{z^{n-1}dz}{(1-z^{-1})(1-az^{-1})}$$

**Basic theorem** $\quad \dfrac{1}{2\pi j}\oint \dfrac{z^{n-1}dz}{1-az^{-1}} = a^n \quad$ for $n \geq 0$

$= 0 \quad$ for n < 0

This allows computation of the integral to be $\boxed{y(n) = \dfrac{1-a^{n+1}}{1-a}}$ $\quad n \geq 0$

This result can be proved by iteration.

## 2. Steady state respond to a complex exponential $e^{jwn}u(n)$

$$y(n) = \sum_{m=0}^{n} h(m)x(n-m) = \sum_{m=0}^{n} x(m)h(n-m)$$

If $x(n) = e^{jwn}u(n)$, then $y(n)$ from above is

$$y(n) = \sum_{m=0}^{n} h(m)e^{jw(n-m)} = e^{jwn}\sum_{m=0}^{n} h(m)e^{-jwm}$$

$$\sum_{m=0}^{n} = \sum_{m=0}^{\infty} - \sum_{m=n+1}^{\infty} \text{, so } y(n) = e^{jwn}\Bigg[\underbrace{\sum_{m=0}^{\infty} h(m)e^{jwn}}_{\text{steady state}} - \underbrace{\sum_{m=n+1}^{\infty} h(m)e^{-jwn}}_{\text{Transmit}}\Bigg]$$

$h(m) \Rightarrow 0$

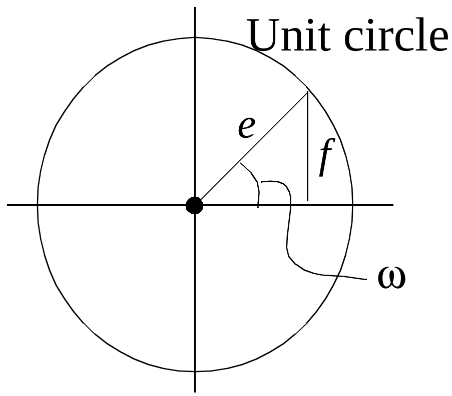
$m \to \infty$, thus sum $\to 0$ as $n \to \infty$

Steady state value of $y(n) = e^{jwn}[H(z)]_{z = e^{jw}}$

So the Frequency response is the value of the z-transform evaluated on the unit circle.

## 3. Geometric Interpretation of Steady State Frequency Response

**for simple first order diff equation :** $H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$




$H(z)$ at $z = e^{jw}$ is $\frac{e}{f}$

## General Rule

Given a collection of poles and zeros in the complex z-plane,

the Frequency response at any $W$ is $\frac{N}{D}$ where $N$ is the product of all vectors to the zeros and $D$ is the product of all vectors to the poles.

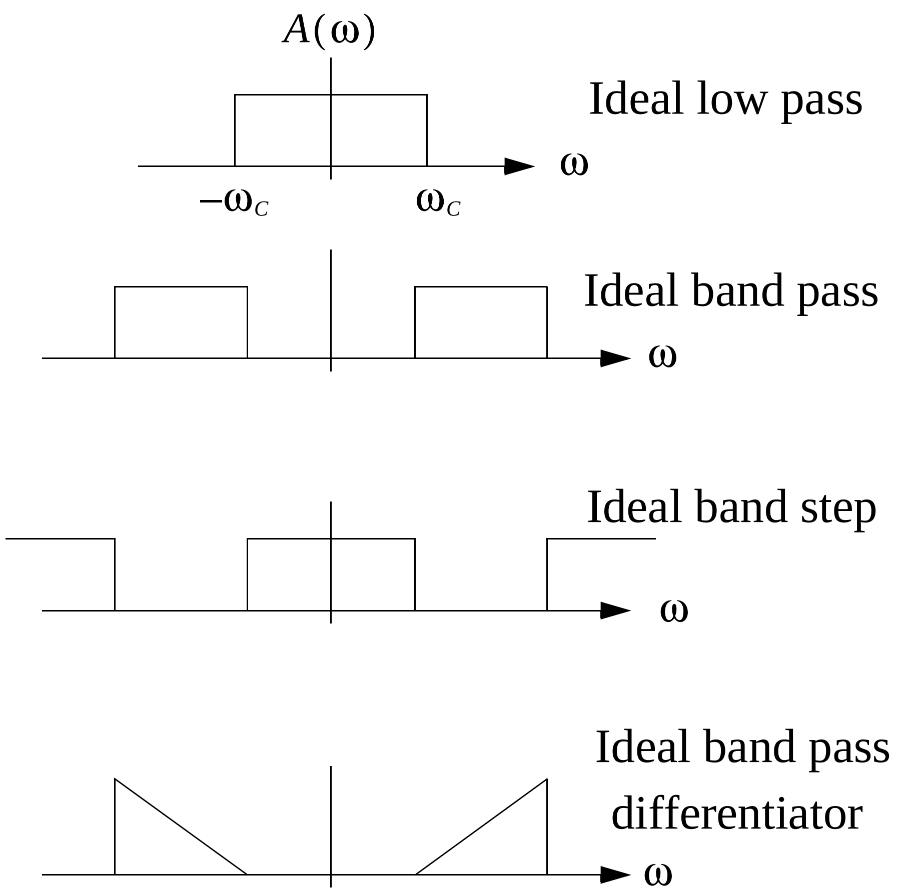
[special rules apply for multiple pole and zeros.]

## Preview of the Rest of the Material

1. Filtering concepts. - approximate problem
2. Sampling and Impulse Invariance
3. Bilinear Transformation
4. The DFT
5. Circular Convolution and Linear Convolution
6. Basic FFT Concept.
7. DFT’s and Filter Banks

## 5. Approximation Problem

Example of Ideal Filters




## Important Point

Linear Analog filters of R, L, C must have frequency responses that are rational functions in $\omega$. Similary, linear digital filter must have rational functions in $e^{j\omega}$

Analog designers tackle the approximation problem by specifying

a REAL function on the jω axis.

$$\text{Example : } |H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

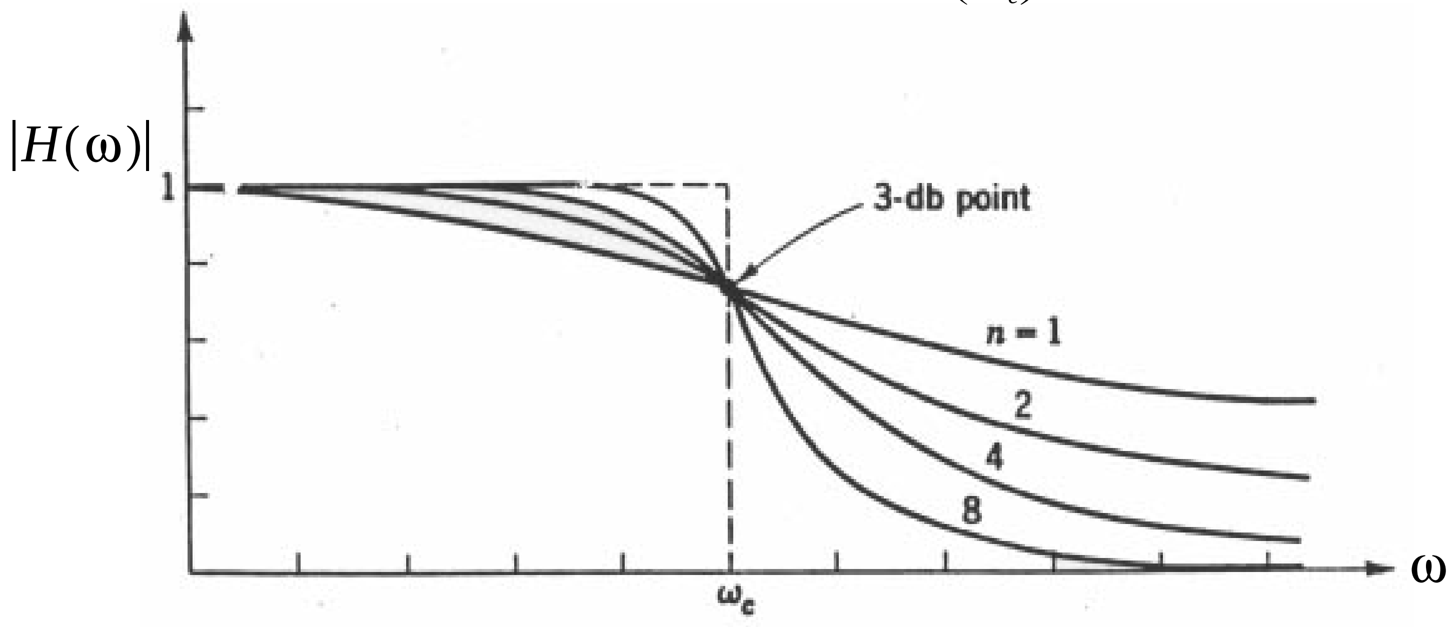



Figure 7.1 : Butterworth Frequency Response for Different n.

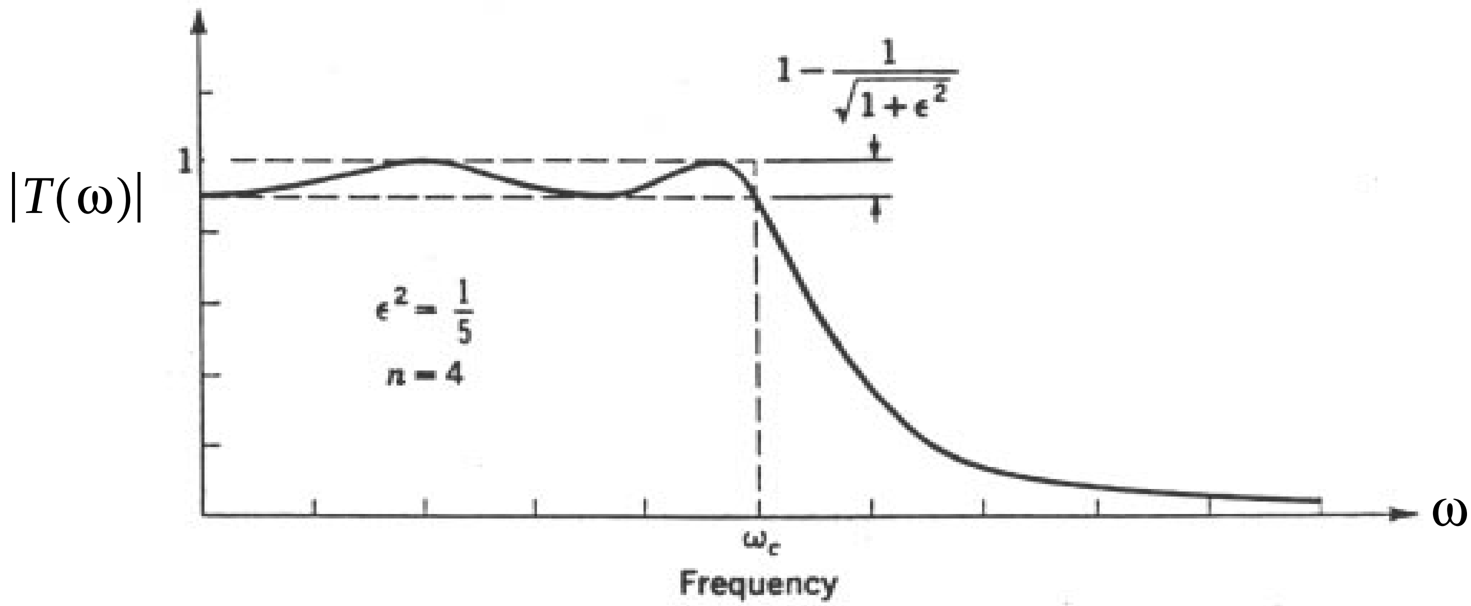



Figure 7.2 : Chebyshev Frequency Response for n=4.

If a suitable $|H(j\omega)|^2$ is chosen, it can lead to a specification in the complex

s-plane of $H(s)$ and this function holds true <u>Everywhere</u> in the s-plane.

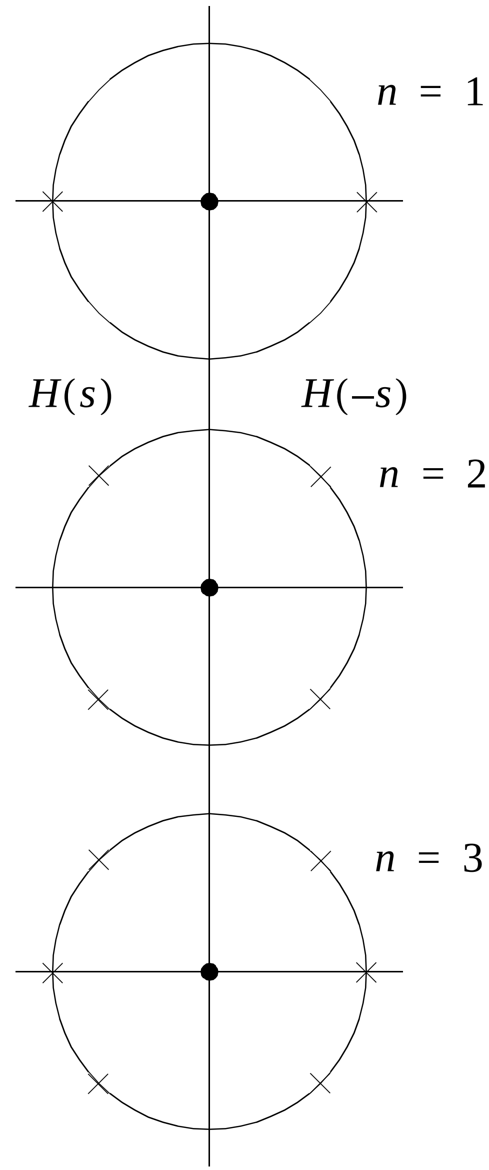
Let's normalize, so that $\gamma = \frac{\omega}{\omega_C}$ and then let $s = j\gamma$ .

So $H(s)H^*(s) = \frac{1}{1 + (-s^2)^n}$

$$H(s)H(-s) = \frac{1}{1 - s^2}$$

$$H(s)H(-s) = \frac{1}{1 + s^4}$$

$$H(s)H(-s) = \frac{1}{1 - s^6}$$



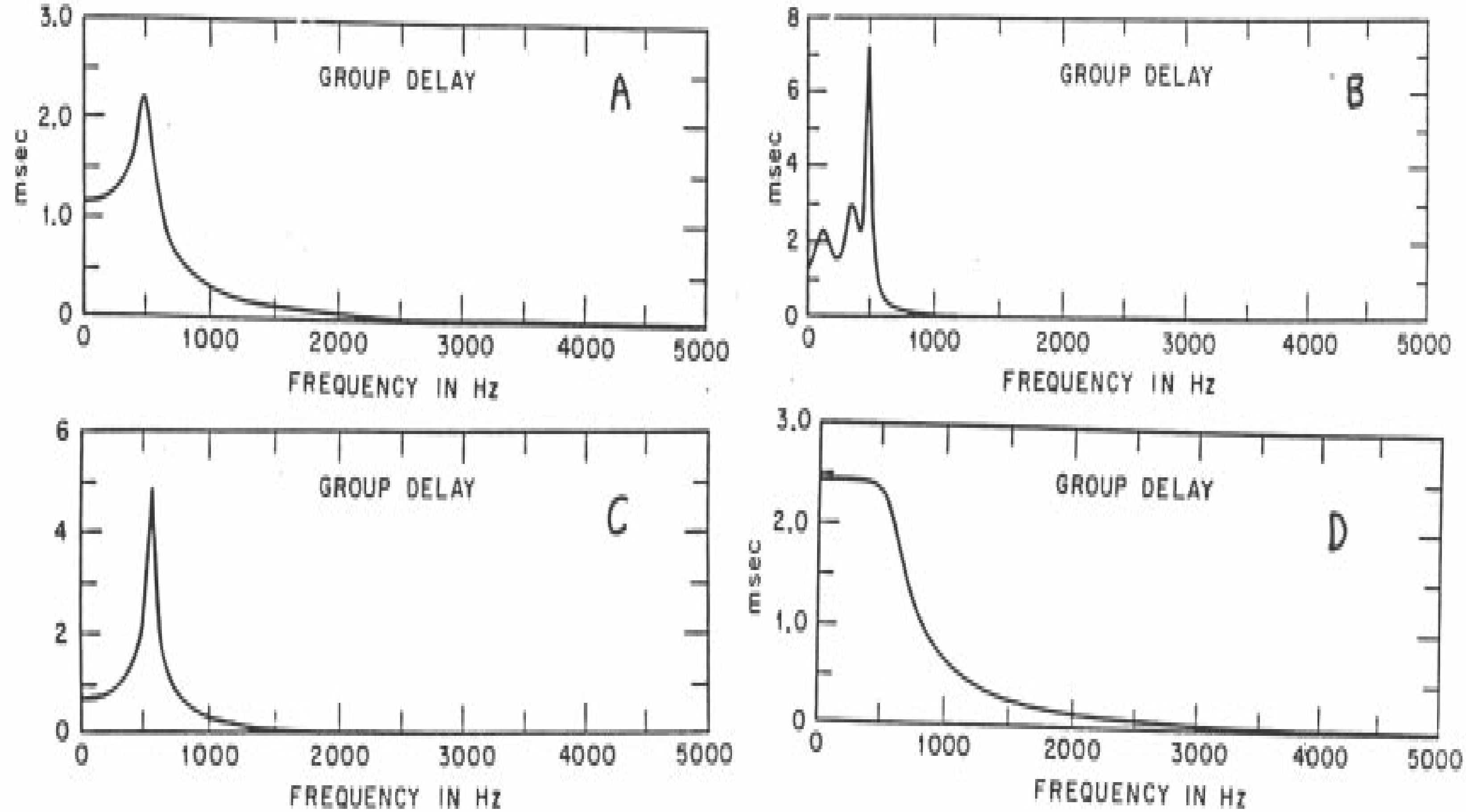



Figure 7.4 : Comparison of Group Delay for Four Types, A = Butterworth, B = Chebyshev, C = Elliptic, D = Bessel, for a low pass filter with a 500Hz corner frequency.

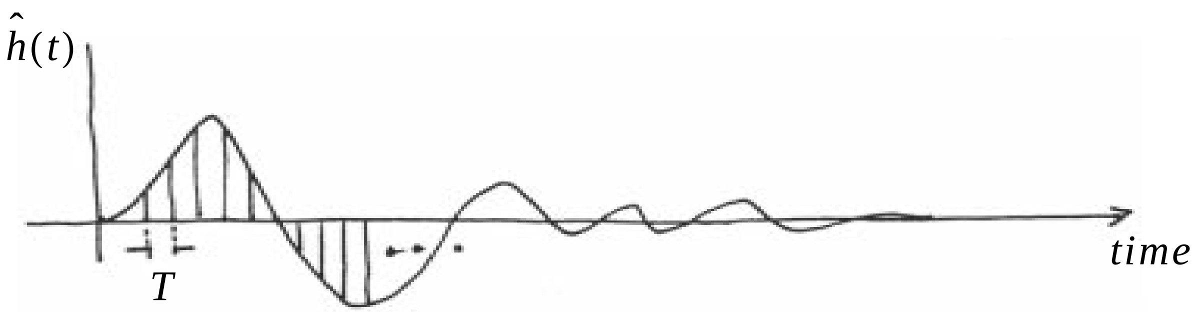
**Raders** channel vocoder experiment - Butterworth filter bank yielded better results than Chebyshev.

Note : Bessel and Lenner filters have good phase response and were used in Vocoders.

**Question :**

How do we construct digital filters that give good frequency responses?

* Impulse Invariance - Linear analog filters have a given impulse response.



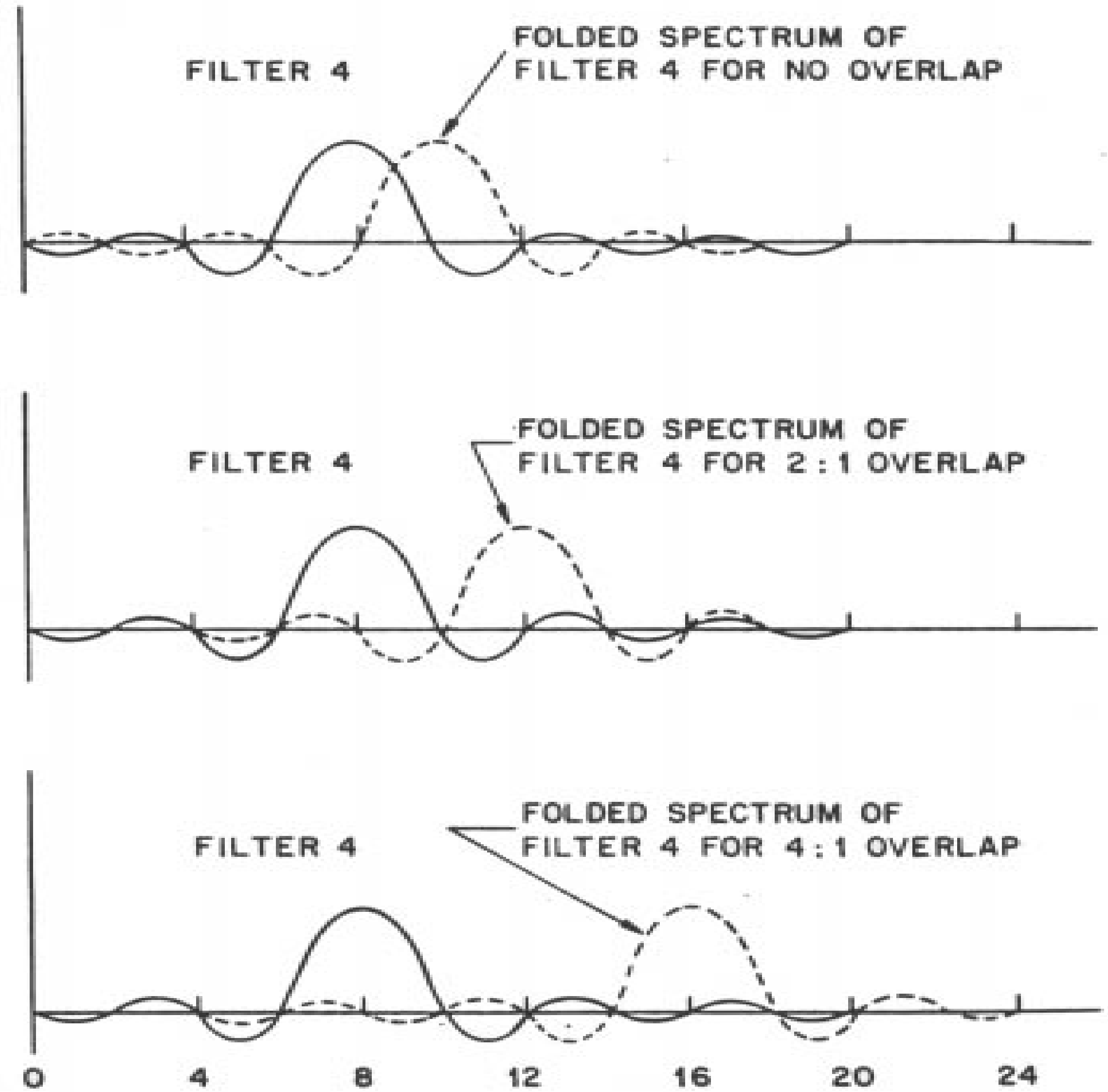


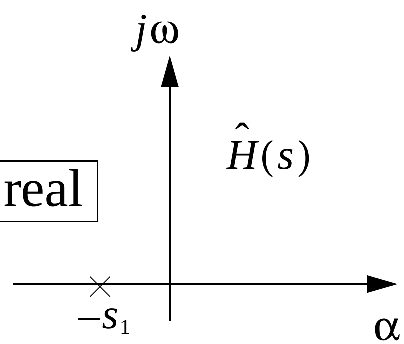

Figure 7.16 : Aliasing Effects of Hopped FFT's.

Construct a <u>Digital Filter</u> that has an impulse response

that are the samples of $h(t)$.

$$h(n) = \hat{h}(nT)$$

$\boxed{s_1 \text{ is real}}$

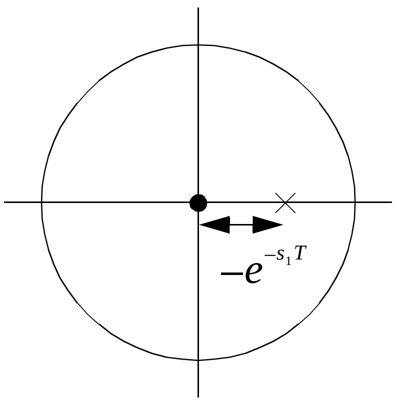



Start with a simple analog filter.

$$\hat{h}(n) = L^{-1}\left(\frac{A_1}{S + S_1}\right) = A_1 e^{-s_1 t}$$

$$h(n) = A_1 e^{-s_1 nT}$$

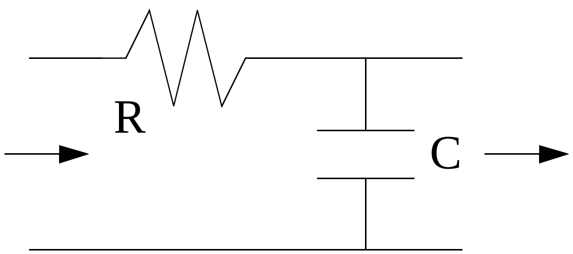
and $$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{A_1}{1 - e^{-s_1 T} z^{-1}}$$



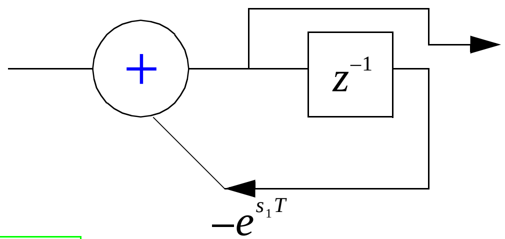
## Procedure

- Find impulse response of suitable analog filter.
- Sample it to find $h(n)$ .
- Take z-transform to find transfer function $H(z)$ .

# Example of Impulse Invariant Design for a Very Simple Case.



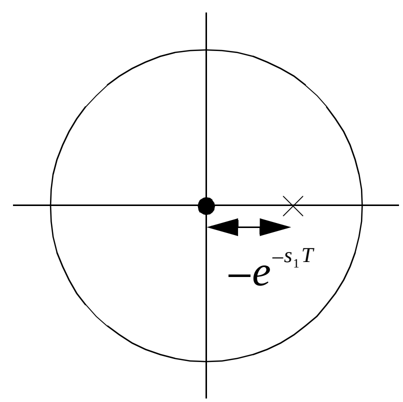

Filter is $\dfrac{1/(sC)}{R + \dfrac{1}{sC}} = \dfrac{1}{1 + sRC}$ and $s_1 = -\dfrac{1}{RC}$




$$y(n) = e^{s_1 T} y(n-1) + x(n)$$

Digital Filter

$$H(z) = \frac{1}{1 - z^{-1} e^{s_1 T}}$$



Aliasing is prevented by using the bilinear transfom to find the digital filter.

$$s \rightarrow \frac{z-1}{z+1}, z = \frac{1+s}{1-s}$$

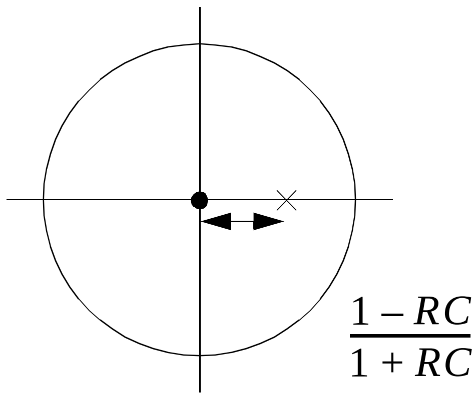
When $s = j\omega$ $z = \frac{1+j\omega}{1-j\omega}$ $\boxed{|z| = 1}$ $j\omega$ axis maps into unit circle.

Stated without proof - Left half s-plane $\Rightarrow$ imterior of z-plane unit circle

Right half s-plane $\Rightarrow$ exterior of z-plane unit circle.

Simple example $\dfrac{1}{1+sRC} \Rightarrow \dfrac{1}{1+\dfrac{z-1}{z+1}RC} = \dfrac{z+1}{z+1+(z-1)RC} = H(z)$

$$H(z) = \frac{z+1}{(1-RC)+z(1+RC)}$$




As $\omega \rightarrow \Pi,\ H(e^{j\omega}) \Rightarrow 0$

No Folding.

For more complex filter designs, multiple zeros appear at $z = -1$.

# Discrete Fourier Transform

Consider a finite duration sequence.

$$X_k = \sum_{n=0}^{n-1} x(n) W^{nk} \qquad W = e^{-j\left(\frac{2\pi}{N}\right)}$$

Inverse $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-nk}$$

Related to z-transform
Related to Fourier transform
Related to Laplace transform
Related to Fourier Series

## Important parameters

Size of DFT N

Size of data.

Window

How often the DFT is done.

Sampling rates.

| $f(nT)$ | $\Leftrightarrow$ | $F(k\Omega)$ |
|---|---|---|
| even | $\Leftrightarrow$ | even |
| odd | $\Leftrightarrow$ | odd |
| even and real | $\Leftrightarrow$ | even and real |
| odd and real | $\Leftrightarrow$ | odd and imaginary |
| real | $\Leftrightarrow$ | real part even<br>imaginary part odd |
| imaginary | $\Leftrightarrow$ | real part odd<br>imaginary part even |
| even and imaginary | $\Leftrightarrow$ | even and imaginary |
| odd and imaginary | $\Leftrightarrow$ | odd and real |

Table 7.1 : Relations between Sequence and its DFT ; recall that an "odd" sequence is antisymmetric, and an "even" sequence is symmetric.

* The DFT can implement an FIR filter exactly.

Because

a) The product of two DFT's corresponds to the circular convolution of two signals

<u>and</u>

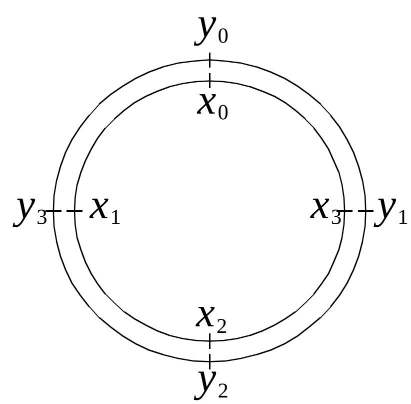
b) By augmenting with zeros, circular convolution can be made equivalent to linear convolution.

# The DFT can Implement Linear Convolution.

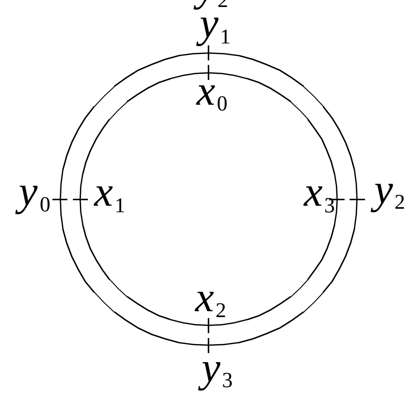
$$X_k = x_0 W^0 + x_1 W^k + x_2 W^{2k} + x_3 W^{3k}$$

$$Y_k = y_0 W^0 + y_1 W^k + y_2 W^{2k} + y_3 W^{3k}$$

$$X_k Y_k = x_0 y_0 + x_1 y_3 + x_2 y_2 + x_3 y_1$$



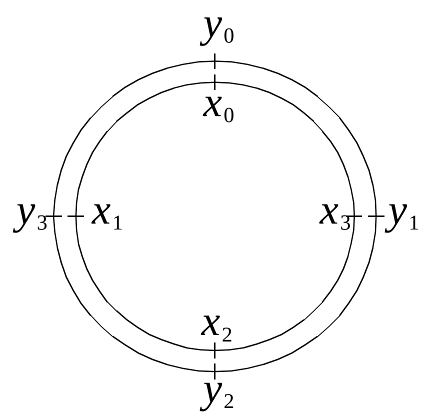

$$+ x_0 y_1 + x_1 y_0 + x_2 y_3 + x_3 y_2$$



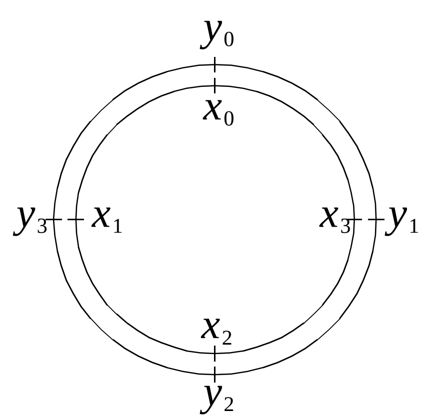

Circular Convolution

# Circular Convolution

$$+\, x_0 y_2 + x_1 y_1 + x_2 y_0 + x_3 y_3$$




$$+\, x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0$$



# Linear Convolution

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | $x_0$ | $x_1$ | $x_2$ | $x_3$ | 0 | 0 | 0 | |
| $x_0 y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | 0 | 0 | | | | |
| $x_0 y_1 + x_1 y_0$ | 0 | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | 0 | 0 | | | |
| $x_0 y_2 + x_1 y_1 + x_2 y_4$ | 0 | 0 | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | 0 | | | |
| | | 0 | 0 | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | | | |
| $\vdots$ | | 0 | 0 | 0 | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | 0 | |
| | | | 0 | 0 | 0 | $y_3$ | $y_2$ | $y_1$ | $y_0$ | 0 | 0 |
| $x_3 y_3$ | | | | | | | $y_3$ | $y_2$ | $y_1$ | $y_0$ | |

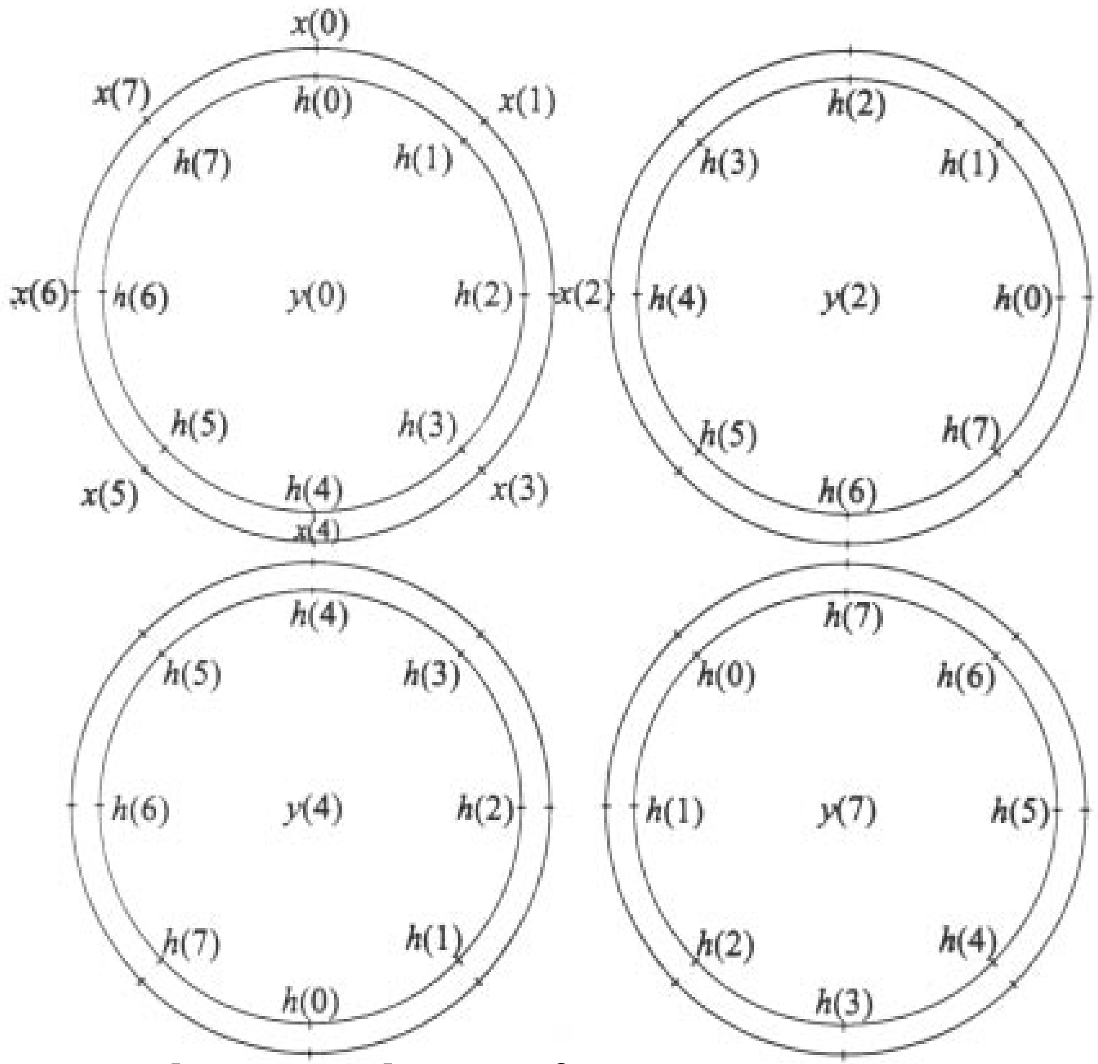



Figure 7.10 : Circular Convolution of Two 8 Point Sequences.
Only y(0), y(2), y(4) and y(7) are shown. All Outer Circles Carry the Same Sequence as the Upper Left Circle.

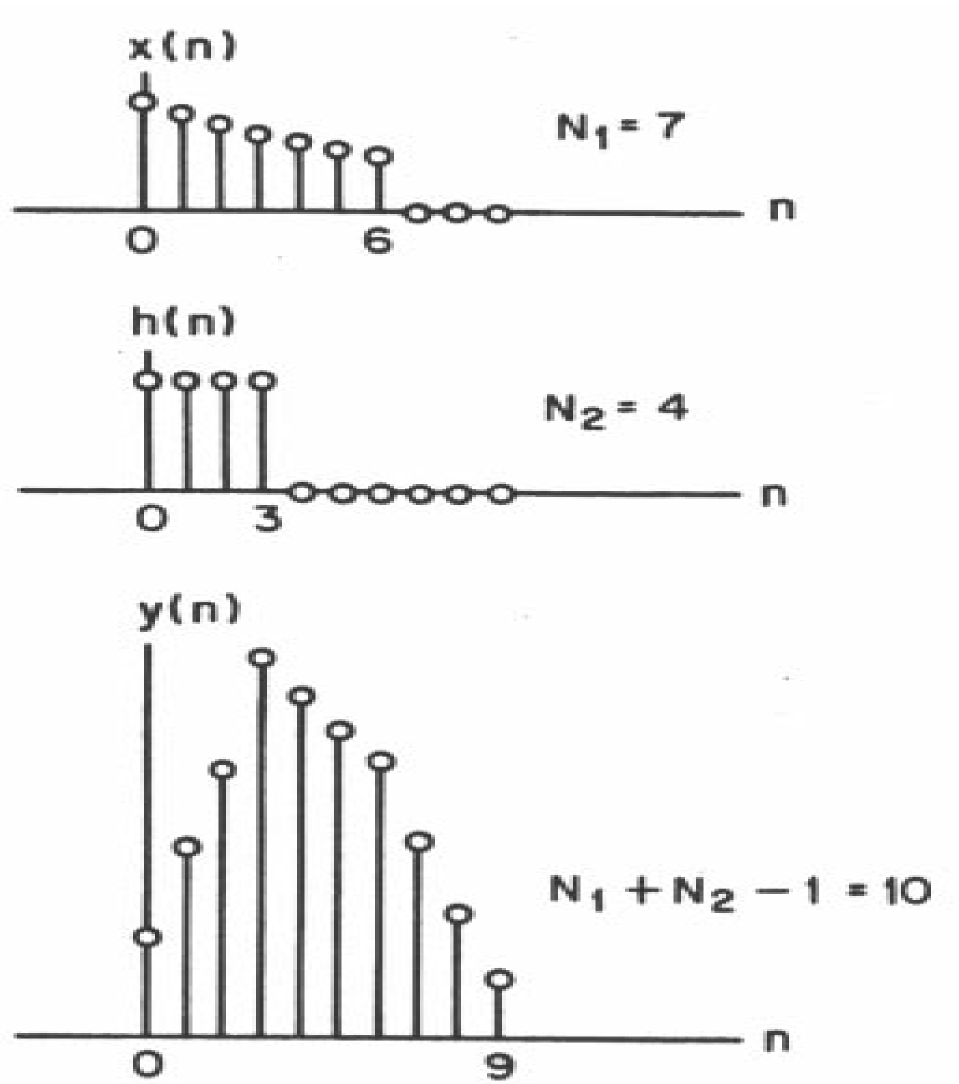



Figure 7.11 : Linear Convolution of Two Finite Length Sequences by DFT.

M

L

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 |
| 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |

$\Downarrow$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.1 | 0,2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.10 | 0.11 |
| 1,0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1,8 | 1.9 | 1.10 | 1.11 |
| 2,0 | 2.1 | 2,2 | 2.3 | 2,4 | 2.5 | 2.6 | 2.7 | 2.8 | 2.9 | 2.10 | 2,11 |
| 3.0 | 3.1 | 3,2 | 3.3 | 3.4 | 3.5 | 3.6 | 3,7 | 3.8 | 3,9 | 3.10 | 3.11 |
| 4.0 | 4.1 | 4,2 | 4.3 | 4.4 | 4.5 | 4.6 | 4.7 | 4,8 | 4.9 | 4.10 | 4.11 |

DFT of each Row - $LM^2$ Operations

Twiddle the Resulting Matrix - LM Operation

DFT of each Column - $ML^2$ Operation

Total $ML(M + L + 1)$

DFT of Complete Array $(ML)^2 = ML(ML)$

$M = 12, \; L = 5$

$M + L + 1 = 18$

$ML = 60$

savings of ~ 3:1

BUT e.g. $M = 1000, \quad L = 20$

$M + L + 1 = 1021$

$ML = 20,000$ savings of ~ 20:1